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Algebraic equations determining quantum dimensions

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Abstract. We show that a quantum dimension $D_q(\Lambda)$ for a representation ρ of $U_q(G)$, a quantized universal enveloping algebra of a compact and simple Lie group G, is computed from the algebraic equations which we found recently in studying $2+1$-dimensional Chern–Simons theory. We solve the equations explicitly for the typical examples of all compact and simple Lie groups. This method can be applied to super Lie groups such as $SU(m,n)$ and $OSp(m,n)$.

Quantum groups [1–4] play important roles in various branches of mathematics and physics (see, for example, [5–13]). However, only a few years have passed since their discovery, and their '*physical*' meaning is not yet clear. It is, therefore, of great value to study '*physical*' aspects of quantum groups. Such investigations may well be useful for grasping a deep understanding of quantum groups.

Quantum groups were discovered in studying exactly soluble models in two dimensions. Rational conformal field theories are known to govern such models. On the other hand, there is a close relationship between $1+1$-dimensional rational conformal field theory and $2+1$-dimensional Chern–Simons theory. The quantum group, therefore, is expected to play an important role in Chern–Simons theory also. Several people are now trying to construct a gauge field theory of a quantum group, in order to make the role of the quantum group clear [14–20]. It is, however, very difficult, and it seems that these approaches still contain conceptual questions.

Recently we found another way of tackling the problem [21]. We constructed algebraic equations satisfied by vacuum expectation values of Wilson loop operators, which are polynomial invariants of coloured knots and links [22–29] in the mathematical literature. This system of equations, however, is over-determined. Namely, the number of equations exceeds the number of variables.

Consequently, consistency amongst such a system is strongly expected to be ensured by some symmetry. We think it must be the quantum group symmetry. Indeed, the vacuum expectation value of an unknotted Wilson loop operator in a representation Λ of a compact and simple Lie group G is nothing but the quantum dimension $D_q(\Lambda)$ of the corresponding representation of $U_q(G)$, a quantized (or Hopf algebra deformation of the) universal enveloping algebra.

In this note we report that $D_q(\Lambda)$'s are really determined by solving the algebraic equations in the case of all compact and simple Lie groups. The purpose of this paper is to stress the existence of such algebraic relations amongst typical quantities of the quantum group, which was found in [21] based upon physical arguments. We believe that the

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comprehensive explicit computations exhibited here must be useful in studying the quantum group itself or related topics.

The equations determining $D_q(\Lambda)$ are summarized in the following proposition.

Proposition. Given decompositions of multiplicity-free tensor products of a finite-dimensional irreducible representation Λ_i with Λ_j, and with its dual $\overline{\Lambda_j}$, of a compact and simple Lie group G:

$$\Lambda_i \otimes \Lambda_j = \bigoplus_{n=1}^{r} \Lambda_n \qquad \text{and} \qquad \Lambda_i \otimes \overline{\Lambda}_j = \bigoplus_{n'=1}^{r'} \Lambda_{n'}.$$

Then, the following algebraic equations for a quantum dimension $D_q(\Lambda)$ of an irreducible represenation Λ of $U_q(G)$, a quantized universal enveloping algebra of G, hold if Λ be integrable ($Q(\Lambda)$ below is a quadratic Casimir of Λ):

$$D_q(\Lambda_i)D_q(\Lambda_j) = \sum_{n=1}^{r} D_q(\Lambda_n) = \sum_{n'=1}^{r'} D_q(\Lambda_{n'}) \tag{1}$$

$$q^{Q(\Lambda_i)+Q(\Lambda_j)} \sum_{n=1}^{r} q^{-Q(\Lambda_n)} D_q(\Lambda_n) = q^{-Q(\Lambda_i)-Q(\Lambda_j)} \sum_{n'=1}^{r'} q^{Q(\Lambda_{n'})} D_q(\Lambda_{n'}) \tag{2}$$

$$q^{-Q(\Lambda_i)-Q(\Lambda_j)} \sum_{n=1}^{r} q^{Q(\Lambda_n)} D_q(\Lambda_n) = q^{Q(\Lambda_i)+Q(\Lambda_j)} \sum_{n'=1}^{r'} q^{-Q(\Lambda_{n'})} D_q(\Lambda_{n'}). \tag{3}$$

If $\Lambda_i = \Lambda_j$, there are two more relations:

$$q^{2Q(\Lambda_i)} D_q(\Lambda_i) = \sum_{n=1}^{r} \beta_n q^{Q(\Lambda_n)/2} D_q(\Lambda_n) \tag{4}$$

$$q^{-2Q(\Lambda_i)} D_q(\Lambda_i) = \sum_{n=1}^{r} \beta_n q^{-Q(\Lambda_n)/2} D_q(\Lambda_n). \tag{5}$$

Here the symmetry factor β_n is $+1$ (-1) if Λ_n is produced as an (anti-) symmetric combination of two Λ_i's. The deformation parameter q, which is a root of unity in this case, is

$$q = \exp\left\{\frac{2\pi \mathrm{i}}{k + Q(\mathrm{Adj})}\right\} \tag{6}$$

where k is an integer (k has the meaning of a level of the affine Lie algebra $\widehat{\mathcal{G}}$ corresponding to the Lie algebra $\mathcal{G}$ of the Lie group G).

This proposition may be amended. For example, it is interesting to investigate whether equations (1)–(5) are also valid to a multiplicity–non-free tensor product. In the following we consider only the case of the tensor products being multiplicity-free. We anticipate, however, the existence of such relations also for multiplicity–non-free cases, although some of equations (1)–(5) might be modified.

The algebraic equations (1)–(5) were constructed upon the basis of physical arguments in [21]. The essential observation is that $D_q(\Lambda)$ has the meaning of a vacuum expectation value of an unknotted Wilson loop operator in the $2+1$-dimensional Chern–Simons theory.

(k is the coupling constant of the theory). Consequently, a close relationship between the Chern–Simons theory and the $1+1$-dimensional rational conformal field theory was used. (From the mathematical point of view $D_q(\Lambda)$ is a Kauffman regular isotopy invariant polynomial for unknots in S^3, normalized so that it is multiplicative for unlinked knots.)

Instead of repeating such a physical argument, we will show in the following that the algebraic relations (1)–(5) can in fact be solved to determine $D_q(\Lambda)$ in the case of typical examples for all compact and simple Lie groups. For the moment we assume that k is sufficiently large. For finite k, as is already discussed in [21], explicit expressions for $D_q(\Lambda)$ calculated below are valid if Λ is the integrable representation†.

Our explicit calculations below strongly support that the algebraic equations (1)–(5) contain enough information to determine the $D_q(\Lambda)$'s, although it is not proved rigorously in this paper. Indeed, it is easily checked explicitly that they offer us too many equations to fix the $D_q(\Lambda)$'s. Consistency amongst the equations must be guaranteed by some symmetry, which is presumably the quantum group.

Knowing the properties of $D_q(\Lambda)$ from, for example, q-deformed character formulae, one may be able to give a rigorous proof of the proposition. Our assertion here, however, resides in a different point: $D_q(\Lambda)$'s are determined iteratively from the algebraic equations (1)–(5) unambiguously. (Note that we require that $D_q(\Lambda)$ becomes the dimension of Λ in the limit that q goes to 1, in order to eliminate one of the two solutions of the quadratic equation.)

$D_q(\Lambda)$ itself is a well investigated quantity. There exist simple and general formulae: lemma 1 of Zhang *et al* [13], for example. Alternative formulae were derived by Wenzl [7] by assigning q-numbers to each box of a Young diagram in the case of $SO(2l+1)$. $D_q(\Lambda)$'s computed in this paper are mainly of the Zhang type, although the Wenzl type of formulae are easily anticipated, at least for $SU(N)$. All of the quantities calculated in this paper agree with the previous results (see also [5, 6, 8–12]).

We think, however, that there are advantages in this note compared with the previous works. Our approach to calculating $D_q(\Lambda)$ by solving algebraic equations is unique, at least to the extent of our knowledge. Moreover, it can be applied for all compact and simple Lie groups. It must be, therefore, helpful and also stimulating in extending, for example, the results of the Wenzl type to $SO(N)$ and $Sp(N)$.

It is not difficult to apply the proposition to super Lie groups such as $SU(m,n)$ and $OSp(m,n)$. But it is not clear whether we are allowed to apply the proposition to such cases, because super-conformal field theories are not yet well understood. (The proposition was derived by exploiting the detailed studies of the compact conformal field theories.) We can, however, expect to get useful information concerning the super-conformal field theories from such calculations. These computations are now in progress [30].

$D_q(\Lambda)$ itself is also important in calculating link polynomials from the so called skein relations. As is well known, $D_q(\Lambda)$, the polynomial for an unknot, is calculated from the skein relations in the case where defining representations of classical Lie groups are assigned to each knot. For other representations, however, $D_q(\Lambda)$ can not be determined from the skein relations, because they contain more than two crossing term(s) [21]. $D_q(\Lambda)$, therefore, must be prepared as inputs for such representations in computing link polynomials by using the skein relations.

† For example, let us label an irreducible representation of $SU(2)$ of dimension d as d. $D_q(d)$ is calculated to be $[d]_{\sqrt{q}}$. This expression, however, is valid if $d \leqslant k+1$ is satisfied, namely if d is an integrable representation, because we required that $D_q(d)$ becomes d under the limit of q going to 1. (Here, $[a]_x = (x^a - x^{-a})/(x - x^{-1})$.)

Table 1. Casimir $Q(\Lambda)$ of Λ_{λ_1}, Λ_{λ_2}, $\Lambda_{2\lambda_1}$, $\Lambda_{\lambda_1+\lambda_2}$, and **1** for $SU(N)$.

Λ	Λ_{λ_1}	Λ_{λ_2}	$\Lambda_{2\lambda_1}$	$\Lambda_{\lambda_1+\lambda_l}$(Adj)	**1**
$Q(\Lambda)$	$\frac{N^2-1}{2N}$	$\frac{N^2-N-2}{N}$	$\frac{N^2+N-2}{N}$	N	0

$A_l = SU(N)$: $N = l+1$. Let us consider the following tensor product decompositions:

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_1} = \Lambda^{+}_{2\lambda_1} \otimes \Lambda^{-}_{\lambda_2}$$

$$\Lambda_{\lambda_1} \otimes \overline{\Lambda}_{\lambda_1} = \Lambda_{\lambda_1+\lambda_l} \otimes \mathbf{1}.$$

Here λ_i, $i = 1, \ldots, l$ are the fundamental weights of A_l. We have used the highest-weight vectors to label the irreducible representations: Λ_{λ_1} for the defining representation, $\Lambda_{\lambda_1+\lambda_l}$ for the adjoint representation, and so on. In general, $\Lambda_{\sum_{k=1}^{l} c_k\lambda_k}$, with non-negative integer c_k, is an irreducible representation whose Young tableau has $\sum_{k=i}^{l} c_k$ boxes in the ith row. ($\overline{\Lambda}_{\sum_{k=1}^{l} c_k\lambda_k} = \Lambda_{\sum_{k=1}^{l} c_k\lambda_{l+1-k}}$ is the conjugate of $\Lambda_{\sum_{k=1}^{l} c_k\lambda_k}$.) The superscript $+$ $(-)$ appearing in the right-hand side of the first equation indicates the (anti-) symmetric combinations of two Λ_{λ_1}'s. $\mathbf{1} = \Lambda_0$ is the identity representation. Algebraic equations constructed from these decompositions are

$$D_q(\Lambda_{\lambda_1})^2 = D_q(\Lambda_{2\lambda_1}) + D_q(\Lambda_{\lambda_2}) = D_q(\Lambda_{\lambda_1+\lambda_l}) + D_q(1)$$

$$q^{\pm 2Q(\Lambda_{\lambda_1})}\left\{q^{\mp Q(\Lambda_{2\lambda_1})}D_q(\Lambda_{2\lambda_1}) + q^{\mp Q(\Lambda_{\lambda_2})}D_q(\Lambda_{\lambda_2})\right\}$$
$$= q^{\mp 2Q(\Lambda_{\lambda_1})}\left\{q^{\pm Q(\Lambda_{\lambda_1+\lambda_l})}D_q(\Lambda_{\lambda_1+\lambda_l}) + q^{\pm Q(1)}D_q(1)\right\}$$

$$q^{\pm 2Q(\Lambda_{\lambda_1})}D_q(\Lambda_{\lambda_1}) = q^{\pm Q(\Lambda_{2\lambda_1})/2}D_q(\Lambda_{2\lambda_1}) - q^{\pm Q(\Lambda_{\lambda_2})}D_q(\Lambda_{\lambda_2}).$$

Here we have used the relation $D_q(\overline{\Lambda}) = D_q(\Lambda)$. It is proved, owing to the property of our algebraic equations being symmetric with respect to $D_q(\overline{\Lambda})$ and $D_q(\Lambda)$. Then, there are six equations for five unknowns†. As we mentioned at the beginning, the consistency of these equations is considered to be guaranteed by the quantum group hidden in the Chern–Simons theory. Indeed, they are solved to yield the following non-trivial solutions by using the quadratic Casimirs given in table 1:

$$D_q(\Lambda_{\lambda_1}) = [N]_{\sqrt{q}}$$

$$D_q(\Lambda_{\lambda_2}) = \frac{[N-1]_{\sqrt{q}}[N]_{\sqrt{q}}}{[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{2\lambda_1}) = \frac{[N+1]_{\sqrt{q}}[N]_{\sqrt{q}}}{[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_l}) = [N+1]_{\sqrt{q}}[N-1]_{\sqrt{q}}$$

$$D_q(1) = 1.$$

They are really the so-called q-dimensions!

† $D_q(\Lambda)D_q(1) = D_q(\Lambda)$ is derived from the decomposition $\Lambda \otimes \mathbf{1} = \Lambda$. We then obtain $D_q(1) = 1$, because we required that $D_q(\Lambda)$ becomes its dimension in the limit $q \to 1$, in order to eliminate redundant solutions in the quadratic equations. There are, therefore, four unknowns in this case. In the following, however, we keep $D_q(1)$ as an unknown variable, and derive $D_q(1) = 1$ from the equations above. This is one of the manifestations of the redundancy residing in our system of algebraic equations.

As a next example, let us consider the following tensor product decompositions:

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} = \Lambda_{\lambda_1+\lambda_2} \oplus \Lambda_{\lambda_3} \qquad \Lambda_{\lambda_1} \otimes \Lambda_{2\lambda_1} = \Lambda_{3\lambda_1} \oplus \Lambda_{\lambda_1+\lambda_2}$$

$$\Lambda_{\lambda_l} \otimes \Lambda_{\lambda_2} = \Lambda_{\lambda_2+\lambda_l} \oplus \Lambda_{\lambda_1} \qquad \Lambda_{\lambda_l} \otimes \Lambda_{2\lambda_1} = \Lambda_{2\lambda_1+\lambda_l} \oplus \Lambda_{\lambda_1}.$$

The algebraic equations derived from these decompositions can be solved easily as

$$D_q(\Lambda_{\lambda_3}) = \frac{[N-2]_{\sqrt{q}}[N-1]_{\sqrt{q}}[N]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_2}) = \frac{[N+1]_{\sqrt{q}}[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}}{[3]_{\sqrt{q}}}$$

$$D_q(\Lambda_{3\lambda_1}) = \frac{[N+2]_{\sqrt{q}}[N+1]_{\sqrt{q}}[N]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{\lambda_2+\lambda_l}) = \frac{[N-2]_{\sqrt{q}}[N]_{\sqrt{q}}[N+1]_{\sqrt{q}}}{[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{2\lambda_1+\lambda_l}) = \frac{[N+2]_{\sqrt{q}}[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}}{[2]_{\sqrt{q}}}.$$

Quadratic Casimirs are listed in table 2.

Table 2. Casimir $Q(\Lambda)$ of Λ_{λ_3}, $\Lambda_{\lambda_1+\lambda_2}$, $\Lambda_{3\lambda_1}$, $\Lambda_{\lambda_2+\lambda_l}$ and $\Lambda_{2\lambda_1+\lambda_l}$ for $SU(N)$.

Λ	Λ_{λ_3}	$\Lambda_{\lambda_1+\lambda_2}$	$\Lambda_{3\lambda_1}$	$\Lambda_{\lambda_2+\lambda_l}$	$\Lambda_{2\lambda_1+\lambda_l}$
$Q(\Lambda)$	$\frac{3N^2-6N-9}{2N}$	$\frac{3N^2-9}{2N}$	$\frac{3N^2+6N-9}{2N}$	$\frac{3N^2-2N-1}{2N}$	$\frac{3N^2+2N-1}{2N}$

Now we consider the following tensor product decompositions:

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_3} = \Lambda_{\lambda_1+\lambda_3} \oplus \Lambda_{\lambda_4}$$

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_1+\lambda_2} = \Lambda_{2\lambda_1+\lambda_2} \oplus \Lambda_{2\lambda_2} \oplus \Lambda_{\lambda_1+\lambda_3}$$

$$\Lambda_{\lambda_1} \otimes \Lambda_{3\lambda_1} = \Lambda_{4\lambda_1} \oplus \Lambda_{2\lambda_1+\lambda_2}$$

$$\Lambda_{\lambda_2} \otimes \Lambda_{\lambda_2} = \Lambda^+_{2\lambda_2} \oplus \Lambda^-_{\lambda_1+\lambda_3} \oplus \Lambda^+_{\lambda_4}$$

$$\Lambda_{2\lambda_1} \otimes \Lambda_{2\lambda_1} = \Lambda^+_{4\lambda_1} \oplus \Lambda^-_{2\lambda_1+\lambda_2} \oplus \Lambda^+_{2\lambda_2}$$

$$\Lambda_{2\lambda_1} \otimes \Lambda_{\lambda_2} = \Lambda_{2\lambda_1+\lambda_2} \oplus \Lambda_{\lambda_1+\lambda_3}.$$

The algebraic equations amongst the $D_q(\Lambda_\lambda)$ with $\lambda = \lambda_4$, $\lambda_1+\lambda_3$, $2\lambda_2$, $2\lambda_1+\lambda_2$ and $4\lambda_1$ are constructed from these decompositions. They are solved to yield the following non-trivial solutions:

$$D_q(\Lambda_{\lambda_4}) = \frac{[N-3]_{\sqrt{q}}[N-2]_{\sqrt{q}}[N-1]_{\sqrt{q}}[N]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_3}) = \frac{[N-2]_{\sqrt{q}}[N-1]_{\sqrt{q}}[N]_{\sqrt{q}}[N+1]_{\sqrt{q}}}{[4]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{2\lambda_2}) = \frac{[N+1]_{\sqrt{q}}[N]_{\sqrt{q}}[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{2\lambda_1+\lambda_2}) = \frac{[N+2]_{\sqrt{q}}[N+1]_{\sqrt{q}}[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}}{[4]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\Lambda_{4\lambda_1}) = \frac{[N+3]_{\sqrt{q}}[N+2]_{\sqrt{q}}[N+1]_{\sqrt{q}}[N]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}.$$

Quadratic Casimirs are listed in table 3. These examples show that $D_q(\Lambda_\lambda)$ is given through the standard method of calculating a dimension of Λ_λ from Young tableaux by assigning not a normal number n but a q-integer $[n]_{\sqrt{q}}$ to each box.

Table 3. Casimir $Q(\Lambda)$ of Λ_{λ_4}, $\Lambda_{\lambda_1+\lambda_3}$, $\Lambda_{2\lambda_2}$, $\Lambda_{2\lambda_1+\lambda_2}$ and $\Lambda_{4\lambda_1}$ for $SU(N)$.

Λ	Λ_{λ_4}	$\Lambda_{\lambda_1+\lambda_3}$	$\Lambda_{2\lambda_2}$	$\Lambda_{2\lambda_1+\lambda_2}$	$\Lambda_{4\lambda_1}$
$Q(\Lambda)$	$\frac{2N^2-6N-8}{N}$	$\frac{2N^2-2N-8}{N}$	$\frac{2N^2-8}{N}$	$\frac{2N^2+2N-8}{N}$	$\frac{2N^2+6N-8}{N}$

The formulae given here are correct for $l \geqslant 4$. For $l = 1$ to 3 we have to introduce the following restrictions

$l = 1$: $\lambda_2 \to 0$ ignore Λ_{λ_3}, $\Lambda_{\lambda_1+\lambda_3}$ and Λ_{λ_4}

$l = 2$: $\lambda_3 \to 0$ ignore Λ_{λ_4}

$l = 3$: $\lambda_4 \to 0$.

Note that for $l = 1$ the exact formuale are easily computed [21]:

$$D_q(\Lambda_{n\lambda_1}) = [n+1]_{\sqrt{q}} \qquad n = 0, 1, 2, \ldots .$$

In the following we shall give some comments in order. Although each $D_q(\Lambda)$ above is given as a ration of a q-integer $[n]_{\sqrt{q}}$, it is a polynomial with respect to q. For example

$$D_q(\Lambda_{\lambda_2}) = \begin{cases} \sum_{m=1}^{(N-1)/2} [4m-1]_{\sqrt{q}} & (N \text{ odd}) \\ \sum_{m=1}^{N/2} [4m-3]_{\sqrt{q}} & (N \text{ even}) \end{cases}$$

$$D_q(\Lambda_{2\lambda_1}) = \begin{cases} \sum_{m=1}^{(N+1)/2} [4m-3]_{\sqrt{q}} & (N \text{ odd}) \\ \sum_{m=1}^{N/2} [4m-1]_{\sqrt{q}} & (N \text{ even}) \end{cases}$$

$$D_q(\Lambda_{\lambda_1+\lambda_l}) = \sum_{m=1}^{N-1} [2m+1]_{\sqrt{q}}\,.$$

(Note that the q-integer $[n]_{\sqrt{q}}$ is a polynomial of q.) Moreover, our calculations show that the maximum absolute value of the exponents of q is always given by

$$(\lambda, \rho).$$

Here $\rho = \sum_{k=1}^{l} \lambda_k$ is half the sum of positive roots.

On the other hand, the parameter q is a root of unity, as shown in equation (6). Consequently, if we require that $D_q(\Lambda_\lambda)$ be expressed as a sum of q-integers $[n]_{q^{1/\alpha}}$ for some integer α, we have to impose the condition

$$\frac{1}{\alpha}\{\alpha(\lambda, \rho) + 1\} < \tfrac{1}{2}\{k + Q(\mathrm{Adj})\} \tag{7}$$

in order to ensure that $D_q(\Lambda_\lambda)$ becomes a dimension of Λ_λ in the limit q goes to 1†.

This condition is satisfied if Λ_λ is the integrable representation. The expressions for the $D_q(\Lambda_\lambda)$'s given in this paper are valid if k is large enough to ensure that Λ_λ is the integrable representation. Note that for the general compact and simple Lie groups the calculations in [21] show

$$\alpha = 2 \times \frac{|\text{long root}|^2}{|\text{short root}|^2} = 2 \times \begin{cases} 1 & \text{for } A_l, D_l, E_6, E_7 \text{ and } E_8 \\ 2 & \text{for } B_l, C_l \text{ and } F_4 \\ 3 & \text{for } G_2. \end{cases}$$

Here the factor 2 appears because of our convention for the deformation parameter q. Under the condition (7) $D_q(\Lambda_\lambda)$ is a positive number satisfying

$$1 \leqslant D_q(\Lambda_\lambda) \leqslant \dim(\Lambda_\lambda).$$

In the appendix we summarize the results of similar computations for other compact and simple Lie groups.

Appendix.

$B_l = SO(N)$: $N = 2l+1$, $l \geqslant 2$ $(N \geqslant 5)$. We consider the following tensor product decompositions:

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_1} = \Lambda^+_{2\lambda_1} \oplus \Lambda^-_{\lambda_2} \oplus \mathbf{1}^+$$

† This condition is derived by using an identity $q^m + q^{-m} = [\alpha m + 1]_{q^{1/\alpha}} - [\alpha m - 1]_{q^{1/\alpha}}$.

$$
\begin{aligned}
&\Lambda_{\lambda_1}\otimes\Lambda_{\lambda_2}=\Lambda_{\lambda_1+\lambda_2}\oplus\Lambda_{\lambda_3}\oplus\Lambda_{\lambda_1}\\
&\Lambda_{\lambda_1}\otimes\Lambda_{2\lambda_1}=\Lambda_{3\lambda_1}\oplus\Lambda_{\lambda_1+\lambda_2}\oplus\Lambda_{\lambda_1}\\
&\Lambda_{\lambda_1}\otimes\Lambda_{\lambda_3}=\Lambda_{\lambda_1+\lambda_3}\oplus\Lambda_{\lambda_4}\oplus\Lambda_{\lambda_2}\\
&\Lambda_{\lambda_1}\otimes\Lambda_{\lambda_1+\lambda_2}=\Lambda_{2\lambda_1+\lambda_2}\oplus\Lambda_{2\lambda_2}\otimes\Lambda_{\lambda_1+\lambda_3}\oplus\Lambda_{2\lambda_1}\oplus\Lambda_{\lambda_2}\\
&\Lambda_{\lambda_1}\otimes\Lambda_{3\lambda_1}=\Lambda_{4\lambda_1}\oplus\Lambda_{2\lambda_1+\lambda_2}\oplus\Lambda_{2\lambda_1}\\
&\Lambda_{\lambda_2}\otimes\Lambda_{\lambda_2}=\Lambda^{+}_{2\lambda_2}\oplus\Lambda^{-}_{\lambda_1+\lambda_3}\oplus\Lambda^{+}_{\lambda_4}\oplus\Lambda^{+}_{2\lambda_1}\oplus\Lambda^{-}_{\lambda_2}\oplus\mathbf{1}^{+}\\
&\Lambda_{2\lambda_1}\otimes\Lambda_{2\lambda_1}=\Lambda^{+}_{4\lambda_1}\oplus\Lambda^{-}_{2\lambda_1+\lambda_2}\oplus\Lambda^{+}_{2\lambda_2}\oplus\Lambda^{+}_{2\lambda_1}\oplus\Lambda^{-}_{\lambda_2}\oplus\mathbf{1}^{+}\\
&\Lambda_{2\lambda_1}\otimes\Lambda_{\lambda_2}=\Lambda_{2\lambda_1+\lambda_2}\oplus\Lambda_{\lambda_1+\lambda_3}\oplus\Lambda_{2\lambda_1}\oplus\Lambda_{\lambda_2}\,.
\end{aligned}
\tag{A1}
$$

Quadratic Casimirs of these representations are listed in table A1. Note that (Al) is valid for $l \geqslant 5$. For small l we have to modify these expressions as

$$
\begin{aligned}
l=2:&\qquad \lambda_2\to 2\lambda_2\quad \lambda_3\to 2\lambda_2\quad \lambda_4\to\lambda_1\\
l=3:&\qquad \lambda_3\to 2\lambda_3\quad \lambda_4\to 2\lambda_3\\
l=4:&\qquad \lambda_4\to 2\lambda_4\,.
\end{aligned}
\tag{A2}
$$

The $Q(\Lambda)$'s listed in table A1 are still valid even after these changes.

Table A1. Casimir $Q(\Lambda)$ of Λ_{λ_1}, Λ_{λ_2}, $\Lambda_{2\lambda_1}$, Λ_{λ_3}, $\Lambda_{\lambda_1+\lambda_2}$, $\Lambda_{3\lambda_1}$, Λ_{λ_4}, $\Lambda_{\lambda_1+\lambda_3}$, $\Lambda_{2\lambda_2}$, $\Lambda_{2\lambda_1+\lambda_2}$ and $\Lambda_{4\lambda_1}$ for $SO(N)$.

Λ	Λ_{λ_1}	Λ_{λ_2}(Adj)	$\Lambda_{2\lambda_1}$	Λ_{λ_3}	$\Lambda_{\lambda_1+\lambda_2}$
$Q(\Lambda)$	$\frac{1}{2}(N-1)$	$N-2$	N	$\frac{1}{2}(3N-9)$	$\frac{1}{2}(3N-3)$
$\Lambda_{3\lambda_1}$	Λ_{λ_4}	$\Lambda_{\lambda_1+\lambda_3}$	$\Lambda_{2\lambda_2}$	$\Lambda_{2\lambda_1+\lambda_2}$	$\Lambda_{4\lambda_1}$
$\frac{1}{2}(3N+3)$	$2N-8$	$2N-4$	$2N-2$	$2N$	$2N+4$

Algebraic equations constructed from these decompositions are easily solved. The results are

$$D_q(\Lambda_{\lambda_1})=\frac{[2N-4]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}\,\frac{[N]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{\lambda_2})=\frac{[2N-8]_{\sqrt[4]{q}}}{[N-4]_{\sqrt[4]{q}}}\,\frac{[2N-2]_{\sqrt[4]{q}}}{[4]_{\sqrt[4]{q}}}\,\frac{[N]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{2\lambda_1})=\frac{[2N-4]_{\sqrt[4]{q}}}{[N-2]_{\sqrt[4]{q}}}\,\frac{[2N-2]_{\sqrt[4]{q}}}{[4]_{\sqrt[4]{q}}}\,\frac{[N+2]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{\lambda_3})=\frac{[2N-12]_{\sqrt[4]{q}}}{[N-6]_{\sqrt[4]{q}}}\,\frac{[2N-2]_{\sqrt[4]{q}}[2N-4]_{\sqrt[4]{q}}}{[6]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}}\,\frac{[N]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_2})=\frac{[2N-8]_{\sqrt[4]{q}}}{[N-4]_{\sqrt[4]{q}}}\,\frac{[2N]_{\sqrt[4]{q}}[2N-4]_{\sqrt[4]{q}}}{[6]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}\,\frac{[N+2]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{3\lambda_1}) = \frac{[2N-4]_{\sqrt[4]{q}}}{[N-2]_{\sqrt[4]{q}}} \frac{[2N]_{\sqrt[4]{q}}[2N-2]_{\sqrt[4]{q}}}{[6]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}} \frac{[N+4]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{\lambda_4}) = \frac{[2N-16]_{\sqrt[4]{q}}}{[N-8]_{\sqrt[4]{q}}} \frac{[2N-2]_{\sqrt[4]{q}}[2N-4]_{\sqrt[4]{q}}[2N-6]_{\sqrt[4]{q}}}{[8]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}} \frac{[N]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_3}) = \frac{[2N-12]_{\sqrt[4]{q}}}{[N-6]_{\sqrt[4]{q}}} \frac{[2N]_{\sqrt[4]{q}}[2N-2]_{\sqrt[4]{q}}[2N-6]_{\sqrt[4]{q}}}{[8]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} \frac{[N+2]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{2\lambda_2}) = \frac{[2N-8]_{\sqrt[4]{q}}}{[N-4]_{\sqrt[4]{q}}} \frac{[2N-4]_{\sqrt[4]{q}}}{[N-2]_{\sqrt[4]{q}}} \frac{[2N+2]_{\sqrt[4]{q}}[2N-6]_{\sqrt[4]{q}}[N+2]_{\sqrt[4]{q}}}{[6]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}} \frac{[N]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{2\lambda_1+\lambda_2}) = \frac{[2N-8]_{\sqrt[4]{q}}}{[N-4]_{\sqrt[4]{q}}} \frac{[2N+2]_{\sqrt[4]{q}}[2N-2]_{\sqrt[4]{q}}[2N-4]_{\sqrt[4]{q}}}{[8]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} \frac{[N+4]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}$$

$$D_q(\Lambda_{4\lambda_1}) = \frac{[2N-4]_{\sqrt[4]{q}}}{[N-2]_{\sqrt[4]{q}}} \frac{[2N+2]_{\sqrt[4]{q}}[2N]_{\sqrt[4]{q}}[2N-2]_{\sqrt[4]{q}}}{[8]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}} \frac{[N+6]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}}.$$

There is a 'spinor' representation Λ_{λ_l} in B_l. It is a self-dual representation and satisfies the following tensor product decomposition:

$$\Lambda_{\lambda_l} \otimes \Lambda_{\lambda_l} = \Lambda^+_{2\lambda_l} \oplus \Lambda^-_{\lambda_{l-1}} \oplus \Lambda^-_{\lambda_{l-2}} \oplus \cdots \oplus \begin{cases} \Lambda^-_{\lambda_1} \oplus \mathbf{1}^- & l = 4n-2 \\ \Lambda^-_{\lambda_1} \oplus \mathbf{1}^+ & l = 4n-1 \\ \Lambda^+_{\lambda_1} \oplus \mathbf{1}^+ & l = 4n \\ \Lambda^+_{\lambda_1} \oplus \mathbf{1}^- & l = 4n+1 \end{cases}$$

where n is a positive integer. The superscripts $+$ and $-$ appear alternately in pairs, except for the first one: $+ - - + + \cdots$. Consequently the $D_q(\Lambda_{\lambda_l})$'s are determined as

$$D_q(\Lambda_{\lambda_l}) = \begin{cases} \left(q^{1/4}+q^{-1/4}\right)\left(q^{3/4}+q^{-3/4}\right) & \text{for } l=2 \\ \left(q^{1/4}+q^{-1/4}\right)\left(q^{3/4}+q^{-3/4}\right)\left(q^{5/4}+q^{-5/4}\right) & \text{for } l=3 \\ \left(q^{1/4}+q^{-1/4}\right)\left(q^{3/4}+q^{-3/4}\right) & \\ \qquad \times \left(q^{5/4}+q^{-5/4}\right)\left(q^{7/4}+q^{-7/4}\right) & \text{for } l=4. \end{cases}$$

From these examples, the following forms for some $D_q(\Lambda)$'s are expected:

$$D_q(\Lambda_{\lambda_i}) = \frac{[2N-4i]_{\sqrt[4]{q}}[N]_{\sqrt[4]{q}}}{[N-2i]_{\sqrt[4]{q}}[2N]_{\sqrt[4]{q}}} \begin{bmatrix} N \\ i \end{bmatrix}_{\sqrt{q}} \qquad \text{for } i = 0, 1, 2, \ldots, l-1$$

$$D_q(\Lambda_{2\lambda_l}) = \frac{[2]_{\sqrt[4]{q}}[N]_{\sqrt[4]{q}}}{[2N]_{\sqrt[4]{q}}} \begin{bmatrix} N \\ l \end{bmatrix}_{\sqrt{q}}$$

$$D_q(\Lambda_{\lambda_l}) = \prod_{n=1}^{l} \left(q^{\frac{2n-1}{4}} + q^{-\frac{2n-1}{4}}\right) = \prod_{n=1}^{l} \frac{[4n-2]_{\sqrt[4]{q}}}{[2n-1]_{\sqrt[4]{q}}}.$$

Here

$$\begin{bmatrix} N \\ i \end{bmatrix}_{\sqrt{q}} = \begin{cases} \dfrac{[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}\cdots[N-i+1]_{\sqrt{q}}}{[i]_{\sqrt{q}}[i-1]_{\sqrt{q}}\cdots[1]_{\sqrt{q}}} & N \geqslant i \geqslant 1 \\ 1 & i = 0. \end{cases}$$

These formulae agree with those given in equation (62) of [13]. Note that quadratic Casimirs are

$$Q\Lambda_{\lambda_i} = \tfrac{1}{2}i(N-i) \qquad \text{for } i = 0, 1, 2, \ldots, l-1$$
$$Q\Lambda_{2\lambda_l} = \tfrac{1}{2}l(N-l) = \tfrac{1}{2}l(l+1).$$

$D_q(\Lambda)$'s for higher spinor representations are also computed. Let us consider the following decompositions:

$$\begin{aligned}
\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_l} &= \Lambda_{\lambda_1+\lambda_l} \oplus \Lambda_{\lambda_l} \\
\Lambda_{\lambda_2} \otimes \Lambda_{\lambda_l} &= \Lambda_{\lambda_2+\lambda_l} \oplus \Lambda_{\lambda_1+\lambda_l} \oplus \Lambda_{\lambda_l} \\
\Lambda_{2\lambda_1} \otimes \Lambda_{\lambda_l} &= \Lambda_{2\lambda_1+\lambda_l} \oplus \Lambda_{\lambda_1+\lambda_l} \\
\Lambda_{\lambda_3} \otimes \Lambda_{\lambda_l} &= \Lambda_{\lambda_3+\lambda_l} \oplus \Lambda_{\lambda_2+\lambda_l} \oplus \Lambda_{\lambda_1+\lambda_l} \oplus \Lambda_{\lambda_l} \\
\Lambda_{\lambda_1+\lambda_2} \otimes \Lambda_{\lambda_l} &= \Lambda_{\lambda_1+\lambda_2+\lambda_l} \oplus \Lambda_{\lambda_2+\lambda_l} \oplus \Lambda_{2\lambda_1+\lambda_l} \oplus \Lambda_{\lambda_1+\lambda_l} \\
\Lambda_{3\lambda_1} \otimes \Lambda_{\lambda_l} &= \Lambda_{3\lambda_1+\lambda_l} \oplus \Lambda_{2\lambda_1+\lambda_l}.
\end{aligned} \tag{A3}$$

The modification (A2) must be taken into account. (Moreover, $\Lambda_{\lambda_3+\lambda_l}$ does not exist for $l = 2$.) Quadratic Casimirs are listed in table A2. From the algebraic equations constructed in this case, $D_q(\Lambda_\lambda)$'s are determined up to† $D_q(\Lambda_{\lambda_l})$:

$$\begin{aligned}
D_q(\Lambda_{\lambda_1+\lambda_l}) &= [N-1]_{\sqrt{q}} D_q(\Lambda_{\lambda_l}) \\
D_q(\Lambda_{\lambda_2+\lambda_l}) &= \frac{[N]_{\sqrt{q}}[N-3]_{\sqrt{q}}}{[2]_{\sqrt{q}}} D_q(\Lambda_{\lambda_l}) \\
D_q(\Lambda_{2\lambda_1+\lambda_l}) &= \frac{[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}}{[2]_{\sqrt{q}}} D_q(\Lambda_{\lambda_l}) \\
D_q(\Lambda_{\lambda_3+\lambda_l}) &= \frac{[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}[N-5]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}} D_q(\Lambda_{\lambda_l}) \\
D_q(\Lambda_{\lambda_1+\lambda_2+\lambda_l}) &= \frac{[N+1]_{\sqrt{q}}[N-1]_{\sqrt{q}}[N-3]_{\sqrt{q}}}{[3]_{\sqrt{q}}} D_q(\Lambda_{\lambda_l}) \\
D_q(\Lambda_{3\lambda_1+\lambda_l}) &= \frac{[N+1]_{\sqrt{q}}[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}} D_q(\Lambda_{\lambda_l})
\end{aligned} \tag{A4}$$

† All of the $D_q(\Lambda_\lambda)$'s are proportional to $D_q(\Lambda_{\lambda_l})$, and so $D_q(\Lambda_{\lambda_l})$ can not be determined from the algebraic equations constructed based upon the decompositions (A3).

Table A2. Casimir $Q(\Lambda)$ of Λ_{λ_l}, $\Lambda_{\lambda_1+\lambda_l}$, $\Lambda_{\lambda_2+\lambda_l}$, $\Lambda_{2\lambda_1+\lambda_l}$, $\Lambda_{\lambda_3+\lambda_l}$, $\Lambda_{\lambda_1+\lambda_2+\lambda_l}$ and $\Lambda_{3\lambda_1+\lambda_l}$ for $SO(N)$.

Λ	Λ_{λ_l}	$\Lambda_{\lambda_1+\lambda_l}$
$Q(\Lambda)$	$\frac{1}{16}(N^2-N)$	$\frac{1}{16}(N^2+7N)$
$\Lambda_{\lambda_2+\lambda_l}$	$\Lambda_{2\lambda_1+\lambda_l}$	$\Lambda_{\lambda_3+\lambda_l}$
$\frac{1}{16}(N^2+15N-16)$	$\frac{1}{16}(N^2+15N+16)$	$\frac{1}{16}(N^2+23N-48)$
$\Lambda_{\lambda_1+\lambda_2+\lambda_l}$	$\Lambda_{3\lambda_1+\lambda_l}$	
$\frac{1}{16}(N^2+23N)$	$\frac{1}{16}(N^2+23N+48)$	

$C_l = Sp(N)$: $N = 2l$, $l \geqslant 3$ $(N \geqslant 6)$. We consider the following tensor product decompositions:

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_1} = \Lambda^+_{2\lambda_1} \oplus \Lambda^-_{\lambda_2} \oplus \mathbf{1}^-$$

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} = \Lambda_{\lambda_1+\lambda_2} \oplus \Lambda_{\lambda_3} \oplus \Lambda_{\lambda_1}$$

$$\Lambda_{\lambda_1} \otimes \Lambda_{2\lambda_1} = \Lambda_{3\lambda_1} \oplus \Lambda_{\lambda_1+\lambda_2} \oplus \Lambda_{\lambda_1}$$

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_3} = \Lambda_{\lambda_1+\lambda_3} \oplus \Lambda_{\lambda_4} \oplus \Lambda_{\lambda_2}$$

$$\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_1+\lambda_2} = \Lambda_{2\lambda_1+\lambda_2} \oplus \Lambda_{2\lambda_2} \oplus \Lambda_{\lambda_1+\lambda_2} \oplus \Lambda_{2\lambda_1} \oplus \Lambda_{\lambda_2}$$

$$\Lambda_{\lambda_1} \otimes \Lambda_{3\lambda_1} = \Lambda_{4\lambda_1} \oplus \Lambda_{2\lambda_1+\lambda_2} \oplus \Lambda_{2\lambda_1}$$

$$\Lambda_{\lambda_2} \otimes \Lambda_{\lambda_2} = \Lambda^+_{2\lambda_2} \oplus \Lambda^-_{\lambda_1+\lambda_3} \oplus \Lambda^+_{\lambda_4} \oplus \Lambda^-_{2\lambda_1} \oplus \Lambda^+_{\lambda_2} \oplus \mathbf{1}^+$$

$$\Lambda_{2\lambda_1} \otimes \Lambda_{2\lambda_1} = \Lambda^+_{4\lambda_1} \oplus \Lambda^-_{2\lambda_1+\lambda_2} \oplus \Lambda^+_{2\lambda_2} \oplus \Lambda^-_{2\lambda_1} \oplus \Lambda^+_{\lambda_2} \oplus \mathbf{1}^+$$

$$\Lambda_{2\lambda_1} \otimes \Lambda_{\lambda_2} = \Lambda_{2\lambda_1+\lambda_2} \oplus \Lambda_{\lambda_1+\lambda_3} \oplus \Lambda_{2\lambda_1} \oplus \Lambda_{\lambda_2}.$$

Quadratic Casimirs are listed in table A3. (Note that Λ_{λ_4} must be deleted for $l = 3$.) Algebraic equations constructed from these decompositions are easily solved. The results are

$$D_q(\Lambda_{\lambda_1}) = \frac{[N+2]_{\sqrt[4]{q}}}{[\frac{1}{2}N+1]_{\sqrt[4]{q}}} [\tfrac{1}{2}N]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{\lambda_2}) = \frac{[N+2]_{\sqrt[4]{q}}}{[\frac{1}{2}N+1]_{\sqrt[4]{q}}} \frac{[N+1]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N-1]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{2\lambda_1}) = \frac{[N+4]_{\sqrt[4]{q}}}{[\frac{1}{2}N+2]_{\sqrt[4]{q}}} \frac{[N+1]_{\sqrt[4]{q}}}{[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{\lambda_3}) = \frac{[N+2]_{\sqrt[4]{q}}}{[\frac{1}{2}N+1]_{\sqrt[4]{q}}} \frac{[N]_{\sqrt[4]{q}}[N+1]_{\sqrt[4]{q}}}{[3]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N-2]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_2}) = \frac{[N+4]_{\sqrt[4]{q}}}{[\frac{1}{2}N+2]_{\sqrt[4]{q}}} \frac{[N]_{\sqrt[4]{q}}[N+2]_{\sqrt[4]{q}}}{[3]_{\sqrt[4]{q}}} [\tfrac{1}{2}N-1]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{3\lambda_1}) = \frac{[N+6]_{\sqrt[4]{q}}}{[\frac{1}{2}N+3]_{\sqrt[4]{q}}} \frac{[N+1]_{\sqrt[4]{q}}[N+2]_{\sqrt[4]{q}}}{[4]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{\lambda_4}) = \frac{[N+2]_{\sqrt[4]{q}}}{[\frac{1}{2}N+1]_{\sqrt[4]{q}}} \frac{[N-1]_{\sqrt[4]{q}}[N]_{\sqrt[4]{q}}[N+1]_{\sqrt[4]{q}}}{[4]_{\sqrt[4]{q}}[3]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N-3]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_3}) = \frac{[N+4]_{\sqrt[4]{q}}}{[\frac{1}{2}N+2]_{\sqrt[4]{q}}} \frac{[N-1]_{\sqrt[4]{q}}[N+1]_{\sqrt[4]{q}}[N+2]_{\sqrt[4]{q}}}{[4]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N-2]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{2\lambda_2}) = \frac{[N+4]_{\sqrt[4]{q}}}{[\frac{1}{2}N+2]_{\sqrt[4]{q}}} \frac{[N+2]_{\sqrt[4]{q}}}{[\frac{1}{2}N+1]_{\sqrt[4]{q}}} \frac{[N-1]_{\sqrt[4]{q}}[N+3]_{\sqrt[4]{q}}}{[3]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N]_{\sqrt[4]{q}}[\tfrac{1}{2}N-1]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{2\lambda_1+\lambda_2}) = \frac{[N+6]_{\sqrt[4]{q}}}{[\frac{1}{2}N+3]_{\sqrt[4]{q}}} \frac{[N]_{\sqrt[4]{q}}[N+1]_{\sqrt[4]{q}}[N+3]_{\sqrt[4]{q}}}{[4]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N-1]_{\sqrt[4]{q}}$$

$$D_q(\Lambda_{4\lambda_1}) = \frac{[N+8]_{\sqrt[4]{q}}}{[\frac{1}{2}N+4]_{\sqrt[4]{q}}} \frac{[N+1]_{\sqrt[4]{q}}[N+2]_{\sqrt[4]{q}}[N+3]_{\sqrt[4]{q}}}{[4]_{\sqrt[4]{q}}[3]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}} [\tfrac{1}{2}N]_{\sqrt[4]{q}}.$$

Table A3. Casimir $Q(\Lambda)$ of Λ_{λ_1}, Λ_{λ_2}, $\Lambda_{2\lambda_1}$, Λ_{λ_3}, $\Lambda_{\lambda_1+\lambda_2}$, $\Lambda_{3\lambda_1}$, Λ_{λ_4}, $\Lambda_{\lambda_1+\lambda_3}$, $\Lambda_{2\lambda_2}$, $\Lambda_{2\lambda_1+\lambda_2}$ and $\Lambda_{4\lambda_1}$ for $Sp(N)$.

Λ	Λ_{λ_1}	Λ_{λ_2}	$\Lambda_{\lambda_{2\lambda_1}}$ (Adj)	Λ_{λ_3}	$\Lambda_{\lambda_1+\lambda_2}$
$Q(\Lambda)$	$\frac{1}{4}(N+1)$	$\frac{1}{2}N$	$\frac{1}{2}(N+2)$	$\frac{1}{4}(3N-3)$	$\frac{1}{4}(3N+3)$
$\Lambda_{3\lambda_1}$	Λ_{λ_4}	$\Lambda_{\lambda_1+\lambda_3}$	$\Lambda_{2\lambda_2}$	$\Lambda_{2\lambda_1+\lambda_2}$	$\Lambda_{4\lambda_1}$
$\frac{1}{4}(3N+9)$	$N-2$	N	$N+1$	$N+2$	$N+4$

$D_l = SO(N)$: $N = 2l$, $l \geqslant 4$ $(N \geqslant 8)$. There is one to one correspondence between ordinary (not spinor) representations of B_l and D_l. The corresponding representations have the same expressions for dimensions and quadratic Casimirs if we use N to express them. They satisfy the same tensor product decompositions (A1). Consequently the $D_q(\Lambda_\lambda)$'s for them coincide with each other. In the case D_l, however, $D_q(\Lambda_\lambda)$'s can be expressed as a rational of, not $[n]_{\sqrt[4]{q}}$, but $[n]_{\sqrt{q}}$, because N is even:

$$D_q(\Lambda_{\lambda_1}) = \frac{[N-2]_{\sqrt{q}}}{[\frac{1}{2}N-1]_{\sqrt{q}}} [\tfrac{1}{2}N]_{\sqrt{q}}$$

$$D_q(\Lambda_{\lambda_2}) = \frac{[N-4]_{\sqrt{q}}}{[\frac{1}{2}N-2]_{\sqrt{q}}} \frac{[N-1]_{\sqrt{q}}}{[2]_{\sqrt{q}}} [\tfrac{1}{2}N]_{\sqrt{q}}$$

$$D_q(\Lambda_{2\lambda_1}) = \frac{[N-2]_{\sqrt{q}}}{[\frac{1}{2}N-1]_{\sqrt{q}}} \frac{[N-1]_{\sqrt{q}}}{[2]_{\sqrt{q}}} [\tfrac{1}{2}N+1]_{\sqrt{q}}$$

$$D_q(\Lambda_{\lambda_3}) = \frac{[N-6]_{\sqrt{q}}}{[\frac{1}{2}N-3]_{\sqrt{q}}} \frac{[N-1]_{\sqrt{q}}[N-2]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}} [\tfrac{1}{2}N]_{\sqrt{q}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_2}) = \frac{[N-4]_{\sqrt{q}}}{[\frac{1}{2}N-2]_{\sqrt{q}}} \frac{[N]_{\sqrt{q}}[N-2]_{\sqrt{q}}}{[3]_{\sqrt{q}}} [\tfrac{1}{2}N+1]_{\sqrt{q}}$$

$$D_q(\Lambda_{3\lambda_1}) = \frac{[N-2]_{\sqrt{q}}}{[\frac{1}{2}N-1]_{\sqrt{q}}} \frac{[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}} [\tfrac{1}{2}N+2]_{\sqrt{q}}$$

$$D_q(\Lambda_{\lambda_4}) = \frac{[N-8]_{\sqrt{q}}}{[\frac{1}{2}N-4]_{\sqrt{q}}} \frac{[N-1]_{\sqrt{q}}[N-2]_{\sqrt{q}}[N-3]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}} [\tfrac{1}{2}N]_{\sqrt{q}}$$

$$D_q(\Lambda_{\lambda_1+\lambda_3}) = \frac{[N-6]_{\sqrt{q}}}{[\frac{1}{2}N-3]_{\sqrt{q}}} \frac{[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}[N-3]_{\sqrt{q}}}{[4]_{\sqrt{q}}[2]_{\sqrt{q}}} [\tfrac{1}{2}N+1]_{\sqrt{q}}$$

$$D_q(\Lambda_{2\lambda_2}) = \frac{[N-4]_{\sqrt{q}}}{[\frac{1}{2}N-2]_{\sqrt{q}}} \frac{[N-2]_{\sqrt{q}}}{[\frac{1}{2}N-1]_{\sqrt{q}}} \frac{[N+1]_{\sqrt{q}}[N-3]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}[2]_{\sqrt{q}}} [\tfrac{1}{2}N+1]_{\sqrt{q}}[\tfrac{1}{2}N]_{\sqrt{q}}$$

$$D_q(\Lambda_{2\lambda_1+\lambda_2}) = \frac{[N-4]_{\sqrt{q}}}{[\frac{1}{2}N-2]_{\sqrt{q}}} \frac{[N+1]_{\sqrt{q}}[N-1]_{\sqrt{q}}[N-2]_{\sqrt{q}}}{[4]_{\sqrt{q}}[2]_{\sqrt{q}}} [\tfrac{1}{2}N+2]_{\sqrt{q}}$$

$$D_q(\Lambda_{4\lambda_1}) = \frac{[N-2]_{\sqrt{q}}}{[\frac{1}{2}N-1]_{\sqrt{q}}} \frac{[N+1]_{\sqrt{q}}[N]_{\sqrt{q}}[N-1]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}} [\tfrac{1}{2}N+3]_{\sqrt{q}}\,.$$

These formulae are valid for $l \geqslant 6$ and the following replacement must be imposed for small l

$$\begin{aligned} l=4:&\qquad \lambda_3 \to \lambda_3+\lambda_4 \qquad \text{and} \qquad \Lambda_{\lambda_4} \to \Lambda_{2\lambda_3} \oplus \Lambda_{2\lambda_4} \\ l=5:&\qquad \lambda_4 \to \lambda_4+\lambda_5. \end{aligned} \tag{A5}$$

Concerning spinor representations, however, there appear differences between B_l and D_l. There are two spinor representations in D_l: $\Lambda_{\lambda_{l-1}}$ and Λ_{λ_l}. They are self-conjugate for even l, conjugate to each other for odd l, and satisfy the following decomposition formulae:

$$\Lambda_{\lambda_{l-1}} \otimes \Lambda_{\lambda_l} = \Lambda_{\lambda_{l-1}+\lambda_l} \oplus \Lambda_{\lambda_{l-3}} \oplus \Lambda_{\lambda_{l-5}} \oplus \cdots \oplus \begin{cases} \Lambda_{\lambda_1} & l \text{ even} \\ 1 & l \text{ odd} \end{cases}$$

$$\Lambda_{\lambda_l} \otimes \Lambda_{\lambda_l} = \Lambda^+_{2\lambda_l} \oplus \Lambda^-_{\lambda_{l-2}} \oplus \Lambda^+_{\lambda_{l-4}} \oplus \cdots \oplus \begin{cases} 1^\pm & l=4n+1\mp 1 \\ \Lambda^\pm_{\lambda_1} & l=4n+2\mp 1 \end{cases}$$

where n is a positive integer. The superscripts $+$ and $-$ appear alternately: $+-+-\cdots$. There is a symmetry between $\Lambda_{\lambda_{l-1}}$ and Λ_{λ_l}, so the second decomposition formula also holds for $\Lambda_{\lambda_{l-1}} \otimes \Lambda_{\lambda_{l-1}}$. We then obtain

$$D_q(\Lambda_{\lambda_{l-1}}) = D_q(\Lambda_{\lambda_l})$$

$$= \begin{cases} \left(q^{1/2}+q^{-1/2}\right)\left(q^{2/2}+q^{-2/2}\right)\left(q^{3/2}+q^{-3/2}\right) & \text{for } l=4 \\ \left(q^{1/2}+q^{-1/2}\right)\left(q^{2/2}+q^{-2/2}\right)\left(q^{3/2}+q^{-3/2}\right) & \\ \qquad \times \left(q^{4/2}+q^{-4/2}\right) & \text{for } l=5 \\ \left(q^{1/2}+q^{-1/2}\right)\left(q^{2/2}+q^{-2/2}\right)\left(q^{3/2}+q^{-3/2}\right) & \\ \qquad \times \left(q^{4/2}+q^{-4/2}\right)\left(q^{5/2}+q^{-5/2}\right) & \text{for } l=6 \end{cases}$$

$$D_q(\Lambda_{\lambda_{l-1}+\lambda_l}) = \begin{cases} \dfrac{[7]_{\sqrt{q}}[6]_{\sqrt{q}}[4]_{\sqrt{q}}}{[3]_{\sqrt{q}}} & \text{for } l=4 \\ \dfrac{[9]_{\sqrt{q}}[8]_{\sqrt{q}}[7]_{\sqrt{q}}[5]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}} & \text{for } l=5 \\ \dfrac{[11]_{\sqrt{q}}[10]_{\sqrt{q}}[9]_{\sqrt{q}}[8]_{\sqrt{q}}[6]_{\sqrt{q}}}{[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}} & \text{for } l=6 \end{cases}$$

$$D_q(\Lambda_{2\lambda_{l-1}}) = D_q(\Lambda_{2\lambda_l}) = \begin{cases} \dfrac{[7]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}}{[3]_{\sqrt{q}}[2]_{\sqrt{q}}} & \text{for } l=4 \\ \dfrac{[9]_{\sqrt{q}}[8]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}} & \text{for } l=5 \\ \dfrac{[11]_{\sqrt{q}}[10]_{\sqrt{q}}[9]_{\sqrt{q}}[8]_{\sqrt{q}}[7]_{\sqrt{q}}}{[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}} & \text{for } l=6. \end{cases}$$

From these examples the following forms for some $D_q(\Lambda_\lambda)$'s are anticipated:

$$D_q(\Lambda_{\lambda_i}) = \frac{[N-2i]_{\sqrt{q}}[\frac{1}{2}N]_{\sqrt{q}}}{[\frac{1}{2}N-i]_{\sqrt{q}}[N]_{\sqrt{q}}}\begin{bmatrix} N \\ i \end{bmatrix}_{\sqrt{q}} \qquad \text{for } i=0,1,2,\ldots,l-2$$

$$D_q(\Lambda_{\lambda_{l-1}+\lambda_l}) = \frac{[2]_{\sqrt{q}}[\frac{1}{2}N]_{\sqrt{q}}}{[N]_{\sqrt{q}}}\begin{bmatrix} N \\ l-1 \end{bmatrix}_{\sqrt{q}}$$

$$D_q(\Lambda_{2\lambda_{l-1}}) = D_q(\Lambda_{2\lambda_l}) = \frac{[\frac{1}{2}N]_{\sqrt{q}}}{[N]_{\sqrt{q}}}\begin{bmatrix} N \\ l \end{bmatrix}_{\sqrt{q}} = \begin{bmatrix} N-1 \\ l-1 \end{bmatrix}_{\sqrt{q}}$$

$$D_q(\Lambda_{\lambda_{l-1}}) = D_q(\Lambda_{\lambda_l}) = \prod_{n=1}^{l-1}(q^{n/2}+q^{-n/2}) = \prod_{n=1}^{l-1}\frac{[2n]_{\sqrt{q}}}{[n]_{\sqrt{q}}}.$$

Note that quadratic Casimirs are:

$$Q(\Lambda_{\lambda_i}) = \tfrac{1}{2}i(N-i) \qquad \text{for } i=0,1,2,\ldots,l-2$$

$$Q(\Lambda_{\lambda_{l-1}+\lambda_l}) = \tfrac{1}{2}(l-1)(N-l+1) = \tfrac{1}{2}(l^2-1)$$

$$Q(\Lambda_{2\lambda_l}) = Q(\Lambda_{2\lambda_{l-1}}) = \tfrac{1}{2}l(N-l) = \tfrac{1}{2}l^2.$$

For higher spinor representations the decompositions (A3) are also satisfied under the modification (A5). Then the $D_q(\Lambda_\lambda)$'s listed in (A4) are also valid. Note that we can replace λ_l with λ_{l-1} in (A3) and (A4) owing to the symmetry between them.

G_2. From now on we use dimensions to distinguish the irreducible representations. Let us consider the following tensor product decompositions:

$$\mathbf{7}\otimes\mathbf{7} = \mathbf{27}_+ \oplus \mathbf{14}_- \oplus \mathbf{7}_- \oplus \mathbf{1}_+$$

$$\mathbf{7}\otimes\mathbf{14} = \mathbf{64} \oplus \mathbf{27} \oplus \mathbf{7}$$

$$\mathbf{7}\otimes\mathbf{27}=\mathbf{77}\oplus\mathbf{64}\oplus\mathbf{27}\oplus\mathbf{14}\oplus\mathbf{7}$$

$$\mathbf{7}\otimes\mathbf{64}=\mathbf{189}\oplus\mathbf{77'}\oplus\mathbf{77}\oplus\mathbf{64}\oplus\mathbf{27}\oplus\mathbf{14}$$

$$\mathbf{7}\otimes\mathbf{77}=\mathbf{189}\oplus\mathbf{182}\oplus\mathbf{77}\oplus\mathbf{64}\oplus\mathbf{27}$$

$$\mathbf{7}\otimes\mathbf{77'}=\mathbf{286}\oplus\mathbf{189}\oplus\mathbf{64}$$

$$\mathbf{14}\otimes\mathbf{14}=\mathbf{77'}_{+}\oplus\mathbf{77}_{-}\oplus\mathbf{27}_{+}\oplus\mathbf{14}_{-}\oplus\mathbf{1}_{+}$$

$$\mathbf{14}\otimes\mathbf{27}=\mathbf{189}\oplus\mathbf{77}\oplus\mathbf{64}\oplus\mathbf{27}\oplus\mathbf{14}\oplus\mathbf{7}$$

$$\mathbf{14}\otimes\mathbf{77}=\mathbf{448}\oplus\mathbf{189}\oplus\mathbf{182}\oplus\mathbf{77'}\oplus\mathbf{77}\oplus\mathbf{64}\oplus\mathbf{27}\oplus\mathbf{14}$$

$$\mathbf{14}\otimes\mathbf{77'}=\mathbf{448}\oplus\mathbf{273}\oplus\mathbf{189}\oplus\mathbf{77'}\oplus\mathbf{77}\oplus\mathbf{14}.$$

7 is the defining representation of G_2. Quadratic Casimirs of these representations are listed in table A4. Then, $D_q(\Lambda_\lambda)$'s are fixed as:

$$D_q(\mathbf{7})=D_q(\Lambda_{\lambda_2})=\frac{[12]_{\sqrt[6]{q}}[7]_{\sqrt[6]{q}}[2]_{\sqrt[6]{q}}}{[6]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{14})=D_q(\Lambda_{\lambda_1})=\frac{[15]_{\sqrt[6]{q}}[8]_{\sqrt[6]{q}}[7]_{\sqrt[6]{q}}}{[5]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}[3]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{27})=D_q(\Lambda_{2\lambda_2})=\frac{[15]_{\sqrt[6]{q}}[12]_{\sqrt[6]{q}}[3]_{\sqrt[6]{q}}}{[5]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{64})=D_q(\Lambda_{\lambda_1+\lambda_2})=\frac{[18]_{\sqrt[6]{q}}[12]_{\sqrt[6]{q}}[10]_{\sqrt[6]{q}}[8]_{\sqrt[6]{q}}[2]_{\sqrt[6]{q}}}{[9]_{\sqrt[6]{q}}[5]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}[3]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{77})=D_q(\Lambda_{3\lambda_2})=\frac{[18]_{\sqrt[6]{q}}[15]_{\sqrt[6]{q}}[11]_{\sqrt[6]{q}}[7]_{\sqrt[6]{q}}}{[9]_{\sqrt[6]{q}}[6]_{\sqrt[6]{q}}[5]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{77'})=D_q(\Lambda_{2\lambda_1})=\frac{[21]_{\sqrt[6]{q}}[12]_{\sqrt[6]{q}}[11]_{\sqrt[6]{q}}[10]_{\sqrt[6]{q}}}{[6]_{\sqrt[6]{q}}[5]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}[3]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{182})=D_q(\Lambda_{4\lambda_2})=\frac{[21]_{\sqrt[6]{q}}[18]_{\sqrt[6]{q}}[13]_{\sqrt[6]{q}}[8]_{\sqrt[6]{q}}}{[9]_{\sqrt[6]{q}}[6]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{189})=D_q(\Lambda_{\lambda_1+2\lambda_2})=\frac{[21]_{\sqrt[6]{q}}[15]_{\sqrt[6]{q}}[12]_{\sqrt[6]{q}}}{[5]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{273})=D_q(\Lambda_{3\lambda_1})=\frac{[27]_{\sqrt[6]{q}}[15]_{\sqrt[6]{q}}[14]_{\sqrt[6]{q}}[13]_{\sqrt[6]{q}}[12]_{\sqrt[6]{q}}}{[9]_{\sqrt[6]{q}}[6]_{\sqrt[6]{q}}[5]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}[3]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{286})=D_q(\Lambda_{2\lambda_1+\lambda_2})=\frac{[24]_{\sqrt[6]{q}}[25]_{\sqrt[6]{q}}[13]_{\sqrt[6]{q}}[11]_{\sqrt[6]{q}}[2]_{\sqrt[6]{q}}}{[6]_{\sqrt[6]{q}}[5]_{\sqrt[6]{q}}[4]_{\sqrt[6]{q}}[3]_{\sqrt[6]{q}}}$$

$$D_q(\mathbf{448})=D_q(\Lambda_{\lambda_1+3\lambda_2})=\frac{[24]_{\sqrt[6]{q}}[28]_{\sqrt[6]{q}}[14]_{\sqrt[6]{q}}[10]_{\sqrt[6]{q}}}{[9]_{\sqrt[6]{q}}[5]_{\sqrt[6]{q}}[3]_{\sqrt[6]{q}}}.$$

Table A4. Casimir $Q(\Lambda)$ of 7, 14, 27, 64, 77, 77′, 182, 189, 273, 286 and 448 for G_2.

Λ	7	14(Adj)	27	64	77	77′	182	189	273	286	448
$Q(\Lambda)$	2	4	$\frac{14}{3}$	7	8	10	12	$\frac{32}{3}$	18	14	15

Here we have also used highest weights to distinguish the representations, as before, for the sake of later convenience.

Through the computations of these examples the general form of $D_q(\Lambda_{m\lambda_1+n\lambda_2})$ is anticipated:

$$D_q(\Lambda_{m\lambda_1+n\lambda_2}) = \frac{[6m+3n+9]_{\sqrt[6]{q}}}{[9]_{\sqrt[6]{q}}} \frac{[3m+3n+6]_{\sqrt[6]{q}}}{[6]_{\sqrt[6]{q}}} \frac{[3m+2n+5]_{\sqrt[6]{q}}}{[5]_{\sqrt[6]{q}}} \times \frac{[3m+n+4]_{\sqrt[6]{q}}}{[4]_{\sqrt[6]{q}}} \frac{[3m+3]_{\sqrt[6]{q}}}{[3]_{\sqrt[6]{q}}} [n+1]_{\sqrt[6]{q}}.$$

Here our convention for the fundamental weights of G_2 is

$$\lambda_1 = \frac{1}{\sqrt{3}}(-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3) \qquad \lambda_2 = \frac{1}{\sqrt{3}}(\varepsilon_2 - \varepsilon_3)$$

with the invariant form $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. The dimension and quadratic Casimir of $\Lambda_{m\lambda_1+n\lambda_2}$ are

$$\dim(\Lambda_{m\lambda_1+n\lambda_2}) = \tfrac{1}{120}(2m+n+3)(m+n+2)(3m+2n+5)(3m+n+4)(m+1)(n+1)$$

$$Q(\Lambda_{m\lambda_1+n\lambda_2}) = m^2 + mn + \tfrac{1}{3}n^2 + 3m + \tfrac{5}{3}n.$$

F_4. We consider the following tensor product decompositions:

$$\mathbf{26} \otimes \mathbf{26} = \mathbf{324}_+ \oplus \mathbf{273}_- \oplus \mathbf{52}_- \oplus \mathbf{26}_+ \oplus \mathbf{1}_+$$

$$\mathbf{26} \otimes \mathbf{52} = \mathbf{1053} \oplus \mathbf{273} \oplus \mathbf{26}$$

$$\mathbf{26} \otimes \mathbf{273} = \mathbf{4096} \oplus \mathbf{1274} \oplus \mathbf{1053} \oplus \mathbf{324} \oplus \mathbf{273} \oplus \mathbf{52} \oplus \mathbf{26}$$

$$\mathbf{26} \otimes \mathbf{324} = \mathbf{4096} \oplus \mathbf{2562} \oplus \mathbf{1053} \oplus \mathbf{324} \oplus \mathbf{273} \oplus \mathbf{26}$$

$$\mathbf{52} \otimes \mathbf{52} = \mathbf{1274}_- \oplus \mathbf{1053}'_+ \oplus \mathbf{324}_+ \oplus \mathbf{52}_- \oplus \mathbf{1}_+$$

$$\mathbf{52} \otimes \mathbf{273} = \mathbf{8424} \oplus \mathbf{4096} \oplus \mathbf{1053} \oplus \mathbf{324} \oplus \mathbf{273} \oplus \mathbf{26}$$

$$\mathbf{52} \otimes \mathbf{324} = \mathbf{10\,829} \oplus \mathbf{4096} \oplus \mathbf{1273} \oplus \mathbf{324} \oplus \mathbf{273} \oplus \mathbf{52}.$$

26 is the defining representation of F_4. Quadratic Casimirs of these representations are list in table A5. Then, the $D_q(\Lambda_\lambda)$'s are determined as

$$D_q(\mathbf{26}) = \frac{[18]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[8]_{\sqrt[4]{q}}[3]_{\sqrt[4]{q}}}{[9]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{52}) = \frac{[20]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[12]_{\sqrt[4]{q}}}{[6]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{273}) = \frac{[20]_{\sqrt[4]{q}}[18]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[7]_{\sqrt[4]{q}}}{[6]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{324}) = \frac{[20]_{\sqrt[4]{q}}[18]_{\sqrt[4]{q}}[15]_{\sqrt[4]{q}}[12]_{\sqrt[4]{q}}[3]_{\sqrt[4]{q}}}{[10]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{1053}) = \frac{[22]_{\sqrt[4]{q}}[18]_{\sqrt[4]{q}}[15]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[12]_{\sqrt[4]{q}}[3]_{\sqrt[4]{q}}}{[11]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{1053'}) = \frac{[24]_{\sqrt[4]{q}}[18]_{\sqrt[4]{q}}[15]_{\sqrt[4]{q}}[14]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[12]_{\sqrt[4]{q}}}{[8]_{\sqrt[4]{q}}[7]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{1273}) = \frac{[22]_{\sqrt[4]{q}}[20]_{\sqrt[4]{q}}[15]_{\sqrt[4]{q}}[14]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[8]_{\sqrt[4]{q}}[7]_{\sqrt[4]{q}}}{[11]_{\sqrt[4]{q}}[10]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}[3]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{2652}) = \frac{[22]_{\sqrt[4]{q}}[20]_{\sqrt[4]{q}}[18]_{\sqrt[4]{q}}[17]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[12]_{\sqrt[4]{q}}}{[11]_{\sqrt[4]{q}}[10]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{4096}) = \frac{[22]_{\sqrt[4]{q}}[20]_{\sqrt[4]{q}}[18]_{\sqrt[4]{q}}[14]_{\sqrt[4]{q}}[12]_{\sqrt[4]{q}}[10]_{\sqrt[4]{q}}[8]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}{[11]_{\sqrt[4]{q}}[9]_{\sqrt[4]{q}}[7]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[3]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{8424}) = \frac{[24]_{\sqrt[4]{q}}[20]_{\sqrt[4]{q}}[18]_{\sqrt[4]{q}}[15]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[12]_{\sqrt[4]{q}}}{[10]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[4]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}$$

$$D_q(\mathbf{10\,829}) = \frac{[24]_{\sqrt[4]{q}}[20]_{\sqrt[4]{q}}[18]_{\sqrt[4]{q}}[17]_{\sqrt[4]{q}}[14]_{\sqrt[4]{q}}[13]_{\sqrt[4]{q}}[10]_{\sqrt[4]{q}}[7]_{\sqrt[4]{q}}[3]_{\sqrt[4]{q}}}{[12]_{\sqrt[4]{q}}[9]_{\sqrt[4]{q}}[8]_{\sqrt[4]{q}}[6]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[5]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}[2]_{\sqrt[4]{q}}}.$$

Table A5. Casimir $Q(\Lambda)$ of **26, 52, 273, 324, 1053, 1053′, 1274, 2652, 4096, 8424** and **10 829** for F_4.

λ	**26**	**52**(Adj)	**273**	**324**	**1053**	**1053′**	**1274**
$Q(\Lambda)$	6	9	12	13	16	20	18
2652	**4096**	**8424**	**10 829**				
21	$\frac{39}{2}$	23	24				

E_6. We consider the following tensor product decompositions:

$$\underline{\mathbf{27}} \otimes \underline{\mathbf{27}} = \overline{\mathbf{351'}}_+ \oplus \overline{\mathbf{351}}_+ \oplus \overline{\mathbf{27}}_+$$

$$\underline{\mathbf{27}} \otimes \overline{\mathbf{27}} = \mathbf{650} \oplus \mathbf{78} \oplus \mathbf{1}$$

$$\underline{\mathbf{27}} \otimes \mathbf{78} = \underline{\mathbf{1728}} \oplus \underline{\mathbf{351}} \oplus \underline{\mathbf{27}}$$

$$\underline{\mathbf{27}} \otimes \underline{\mathbf{351}} = \underline{\mathbf{7371}} \oplus \mathbf{1728} \oplus \mathbf{351} \oplus \mathbf{27}$$

$$\underline{\mathbf{27}} \otimes \overline{\mathbf{351}} = \underline{\mathbf{5824}} \oplus \mathbf{2925} \oplus \mathbf{650} \oplus \mathbf{78}$$

$$\underline{\mathbf{27}} \otimes \underline{\mathbf{351'}} = \overline{\mathbf{7722}} \oplus \overline{\mathbf{1728}} \oplus \overline{\mathbf{27}}$$

$$\underline{\mathbf{27}} \otimes \overline{\mathbf{351'}} = \underline{\mathbf{5824}} \oplus \underline{\mathbf{3003}} \oplus \mathbf{650}$$

$$\mathbf{78} \otimes \mathbf{78} = \mathbf{2925}_- \oplus \mathbf{2430}_+ \oplus \mathbf{650}_+ \oplus \mathbf{78}_- \oplus \mathbf{1}_+ .$$

27 is the defining representation of E_6. Quadratic Casimirs of these representations are listed in table A6. Then, the $D_q(\Lambda_\lambda)$'s are computed as:

$$D_q(\mathbf{27}) = \frac{[12]_{\sqrt{q}}[9]_{\sqrt{q}}}{[4]_{\sqrt{q}}}$$

$$D_q(\mathbf{78}) = \frac{[13]_{\sqrt{q}}[9]_{\sqrt{q}}[8]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}}$$

$$D_q(\underline{\mathbf{351}}) = \frac{[13]_{\sqrt{q}}[12]_{\sqrt{q}}[9]_{\sqrt{q}}[6]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\underline{\mathbf{351}'}) = \frac{[13]_{\sqrt{q}}[12]_{\sqrt{q}}[10]_{\sqrt{q}}[9]_{\sqrt{q}}}{[5]_{\sqrt{q}}[4]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\mathbf{650}) = \frac{[13]_{\sqrt{q}}[12]_{\sqrt{q}}[10]_{\sqrt{q}}[8]_{\sqrt{q}}[5]_{\sqrt{q}}}{[6]_{\sqrt{q}}[4]_{\sqrt{q}}[4]_{\sqrt{q}}}$$

$$D_q(\underline{\mathbf{1728}}) = \frac{[14]_{\sqrt{q}}[12]_{\sqrt{q}}[10]_{\sqrt{q}}[9]_{\sqrt{q}}[8]_{\sqrt{q}}[6]_{\sqrt{q}}}{[7]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}}$$

$$D_q(\mathbf{2430}) = \frac{[15]_{\sqrt{q}}[12]_{\sqrt{q}}[10]_{\sqrt{q}}[9]_{\sqrt{q}}[9]_{\sqrt{q}}[8]_{\sqrt{q}}}{[5]_{\sqrt{q}}[4]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\mathbf{2925}) = \frac{[14]_{\sqrt{q}}[13]_{\sqrt{q}}[10]_{\sqrt{q}}[9]_{\sqrt{q}}[9]_{\sqrt{q}}[5]_{\sqrt{q}}}{[7]_{\sqrt{q}}[3]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\underline{\mathbf{3003}}) = \frac{[14]_{\sqrt{q}}[13]_{\sqrt{q}}[12]_{\sqrt{q}}[11]_{\sqrt{q}}[10]_{\sqrt{q}}[9]_{\sqrt{q}}}{[6]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\underline{\mathbf{5824}}) = \frac{[14]_{\sqrt{q}}[13]_{\sqrt{q}}[12]_{\sqrt{q}}[10]_{\sqrt{q}}[8]_{\sqrt{q}}[6]_{\sqrt{q}}}{[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}[3]_{\sqrt{q}}}$$

$$D_q(\underline{\mathbf{7371}}) = \frac{[14]_{\sqrt{q}}[13]_{\sqrt{q}}[12]_{\sqrt{q}}[9]_{\sqrt{q}}[9]_{\sqrt{q}}}{[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\underline{\mathbf{7722}}) = \frac{[14]_{\sqrt{q}}[13]_{\sqrt{q}}[12]_{\sqrt{q}}[11]_{\sqrt{q}}[9]_{\sqrt{q}}[8]_{\sqrt{q}}}{[7]_{\sqrt{q}}[4]_{\sqrt{q}}[4]_{\sqrt{q}}[2]_{\sqrt{q}}}.$$

Table A6. Casimir $Q(\Lambda)$ of **27**, **78**, **351**, **351′**, **650**, **1728**, **2430**, **2925**, **3003**, **5824**, **7371** and **7722** for E_6.

Λ	**27**	**78**(Adj)	**351**	**351′**	**650**	**1728**	**2430**	**2925**
$Q(\Lambda)$	$\frac{26}{3}$	12	$\frac{50}{3}$	$\frac{56}{3}$	18	$\frac{65}{3}$	26	24
3003	**5824**	**7371**	**7722**					
30	27	$\frac{80}{3}$	$\frac{86}{3}$					

E_7. We consider the following tensor product decompositions:

$$\mathbf{56}\otimes\mathbf{56}=\mathbf{1539}_-\oplus\mathbf{1463}_+\oplus\mathbf{133}_+\oplus\mathbf{1}_-$$

$$\mathbf{56}\otimes\mathbf{133}=\mathbf{6480}\oplus\mathbf{912}\oplus\mathbf{56}$$

$$\mathbf{56}\otimes\mathbf{912}=\mathbf{40\,755}\oplus\mathbf{8645}\oplus\mathbf{1539}\oplus\mathbf{133}$$

$$\mathbf{56}\otimes\mathbf{1463}=\mathbf{51\,072}\oplus\mathbf{24\,320}\oplus\mathbf{6480}\oplus\mathbf{56}$$

$$\mathbf{56}\otimes\mathbf{1539}=\mathbf{51\,072}\oplus\mathbf{27\,664}\oplus\mathbf{6480}\oplus\mathbf{912}\oplus\mathbf{56}$$

$$\mathbf{133}\otimes\mathbf{133}=\mathbf{8645}_-\oplus\mathbf{7371}_+\oplus\mathbf{1539}_+\oplus\mathbf{133}_-\oplus\mathbf{1}_+$$

$$\mathbf{133}\otimes\mathbf{912}=\mathbf{86\,184}\oplus\mathbf{27\,664}\oplus\mathbf{6480}\oplus\mathbf{912}\oplus\mathbf{56}\,.$$

56 is the defining representation of E_7. Quadratic Casimirs of these representations are listed in table A7. Then, the $D_q(\Lambda_\lambda)$'s are calculated as

$$D_q(\mathbf{56})=\frac{[18]_{\sqrt{q}}[14]_{\sqrt{q}}[10]_{\sqrt{q}}}{[9]_{\sqrt{q}}[5]_{\sqrt{q}}}$$

$$D_q(\mathbf{133})=\frac{[19]_{\sqrt{q}}[14]_{\sqrt{q}}[12]_{\sqrt{q}}}{[6]_{\sqrt{q}}[4]_{\sqrt{q}}}$$

$$D_q(\mathbf{912})=\frac{[19]_{\sqrt{q}}[18]_{\sqrt{q}}[14]_{\sqrt{q}}[10]_{\sqrt{q}}[8]_{\sqrt{q}}}{[7]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}}$$

$$D_q(\mathbf{1463})=\frac{[19]_{\sqrt{q}}[18]_{\sqrt{q}}[15]_{\sqrt{q}}[14]_{\sqrt{q}}[11]_{\sqrt{q}}}{[9]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\mathbf{1539})=\frac{[19]_{\sqrt{q}}[18]_{\sqrt{q}}[15]_{\sqrt{q}}[12]_{\sqrt{q}}}{[5]_{\sqrt{q}}[4]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\mathbf{6480})=\frac{[20]_{\sqrt{q}}[18]_{\sqrt{q}}[15]_{\sqrt{q}}[14]_{\sqrt{q}}[12]_{\sqrt{q}}}{[7]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}}$$

$$D_q(\mathbf{7371})=\frac{[21]_{\sqrt{q}}[18]_{\sqrt{q}}[15]_{\sqrt{q}}[14]_{\sqrt{q}}[13]_{\sqrt{q}}[12]_{\sqrt{q}}}{[7]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\mathbf{8645})=\frac{[20]_{\sqrt{q}}[19]_{\sqrt{q}}[15]_{\sqrt{q}}[14]_{\sqrt{q}}[13]_{\sqrt{q}}}{[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\mathbf{24\,320})=\frac{[20]_{\sqrt{q}}[19]_{\sqrt{q}}[18]_{\sqrt{q}}[16]_{\sqrt{q}}[15]_{\sqrt{q}}[14]_{\sqrt{q}}[12]_{\sqrt{q}}}{[9]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\mathbf{27\,664})=\frac{[20]_{\sqrt{q}}[19]_{\sqrt{q}}[18]_{\sqrt{q}}[14]_{\sqrt{q}}[13]_{\sqrt{q}}[12]_{\sqrt{q}}[6]_{\sqrt{q}}}{[9]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(\mathbf{40\,755})=\frac{[20]_{\sqrt{q}}[19]_{\sqrt{q}}[18]_{\sqrt{q}}[15]_{\sqrt{q}}[13]_{\sqrt{q}}[11]_{\sqrt{q}}}{[6]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}}$$

$$D_q(51\,072) = \frac{[20]_{\sqrt{q}}[19]_{\sqrt{q}}[18]_{\sqrt{q}}[16]_{\sqrt{q}}[14]_{\sqrt{q}}[12]_{\sqrt{q}}}{[6]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}}$$

$$D_q(86\,184) = \frac{[21]_{\sqrt{q}}[19]_{\sqrt{q}}[18]_{\sqrt{q}}[15]_{\sqrt{q}}[14]_{\sqrt{q}}[12]_{\sqrt{q}}[10]_{\sqrt{q}}}{[7]_{\sqrt{q}}[5]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}}.$$

Table A7. Casimir $Q(\Lambda)$ of 56, 133, 912, 1463, 1539, 6480, 7371, 8645, 24 320, 27 664, 40 755, 51 072 and 86 184 for E_8.

Λ	56	133(Adj)	912	1463	1539	6480	7371
$Q(\Lambda)$	$\frac{57}{4}$	18	$\frac{105}{4}$	30	28	$\frac{133}{4}$	38
8645	24 320	27 664	40 755	51 072	86 184		
36	$\frac{189}{4}$	$\frac{165}{4}$	42	$\frac{177}{4}$	$\frac{185}{4}$		

E_8. We consider the following tensor product decompositions:

$$\mathbf{248} \otimes \mathbf{248} = \mathbf{30\,380}_- \oplus \mathbf{27\,000}_+ \oplus \mathbf{3875}_+ \oplus \mathbf{248}_- \oplus \mathbf{1}_+$$

$$\mathbf{248} \otimes \mathbf{3875} = \mathbf{779\,247} \oplus \mathbf{147\,250} \oplus \mathbf{30\,380} \oplus \mathbf{3875} \oplus \mathbf{248}$$

$$\mathbf{248} \otimes \mathbf{27\,000} = \mathbf{4\,096\,000} \oplus \mathbf{1\,763\,125} \oplus \mathbf{779\,247} \oplus \mathbf{30\,380} \oplus \mathbf{27\,000} \oplus \mathbf{248}$$

$$\mathbf{248} \otimes \mathbf{30\,380} = \mathbf{4\,096\,000} \oplus \mathbf{2\,450\,240} \oplus \mathbf{779\,247} \oplus \mathbf{147\,250} \oplus \mathbf{30\,380} \oplus \mathbf{27\,000} \oplus \mathbf{3875} \oplus \mathbf{248}$$

$$\mathbf{3875} \otimes \mathbf{3875} = \mathbf{6\,696\,000}_- \oplus \mathbf{4\,881\,384}_+ \oplus \mathbf{2\,450\,240}_+ \oplus \mathbf{779\,247}_- \oplus \mathbf{147\,250}_+ \oplus \mathbf{30\,380}_- \oplus \mathbf{27\,000}_+ \oplus \mathbf{3875}_+ \oplus \mathbf{248}_- \oplus \mathbf{1}_+.$$

248 is the defining and simultaneously the adjoint representation of E_8. Quadratic Casimirs of these representations are listed in table A8. Then, the $D_q(\Lambda_\lambda)$'s are determined as

$$D_q(248) = \frac{[31]_{\sqrt{q}}[24]_{\sqrt{q}}[20]_{\sqrt{q}}}{[10]_{\sqrt{q}}[6]_{\sqrt{q}}}$$

$$D_q(3875) = \frac{[31]_{\sqrt{q}}[30]_{\sqrt{q}}[25]_{\sqrt{q}}[20]_{\sqrt{q}}[14]_{\sqrt{q}}}{[10]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}[4]_{\sqrt{q}}}$$

$$D_q(27\,000) = \frac{[33]_{\sqrt{q}}[30]_{\sqrt{q}}[25]_{\sqrt{q}}[24]_{\sqrt{q}}[21]_{\sqrt{q}}[20]_{\sqrt{q}}}{[11]_{\sqrt{q}}[10]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(30\,380) = \frac{[32]_{\sqrt{q}}[31]_{\sqrt{q}}[25]_{\sqrt{q}}[24]_{\sqrt{q}}[21]_{\sqrt{q}}[18]_{\sqrt{q}}[14]_{\sqrt{q}}}{[16]_{\sqrt{q}}[12]_{\sqrt{q}}[9]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(147\,250) = \frac{[32]_{\sqrt{q}}[31]_{\sqrt{q}}[30]_{\sqrt{q}}[25]_{\sqrt{q}}[21]_{\sqrt{q}}[20]_{\sqrt{q}}[19]_{\sqrt{q}}}{[16]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}}$$

$$D_q(779\,247) = \frac{[33]_{\sqrt{q}}[31]_{\sqrt{q}}[30]_{\sqrt{q}}[26]_{\sqrt{q}}[24]_{\sqrt{q}}[21]_{\sqrt{q}}[19]_{\sqrt{q}}[14]_{\sqrt{q}}[12]_{\sqrt{q}}}{[13]_{\sqrt{q}}[11]_{\sqrt{q}}[10]_{\sqrt{q}}[8]_{\sqrt{q}}[6]_{\sqrt{q}}[6]_{\sqrt{q}}[4]_{\sqrt{q}}}$$

$$D_q(1\,763\,125) = \frac{[35]_{\sqrt{q}}[31]_{\sqrt{q}}[30]_{\sqrt{q}}[26]_{\sqrt{q}}[25]_{\sqrt{q}}[24]_{\sqrt{q}}[22]_{\sqrt{q}}[21]_{\sqrt{q}}[20]_{\sqrt{q}}}{[12]_{\sqrt{q}}[11]_{\sqrt{q}}[10]_{\sqrt{q}}[8]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(2\,450\,240) = \frac{[33]_{\sqrt{q}}[32]_{\sqrt{q}}[31]_{\sqrt{q}}[26]_{\sqrt{q}}[25]_{\sqrt{q}}[24]_{\sqrt{q}}[20]_{\sqrt{q}}[19]_{\sqrt{q}}}{[11]_{\sqrt{q}}[10]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(4\,096\,000) = \frac{[34]_{\sqrt{q}}[32]_{\sqrt{q}}[30]_{\sqrt{q}}[26]_{\sqrt{q}}[25]_{\sqrt{q}}[24]_{\sqrt{q}}[22]_{\sqrt{q}}[20]_{\sqrt{q}}[18]_{\sqrt{q}}[14]_{\sqrt{q}}}{[17]_{\sqrt{q}}[13]_{\sqrt{q}}[11]_{\sqrt{q}}[9]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[3]_{\sqrt{q}}}$$

$$D_q(4\,881\,384) = \frac{[33]_{\sqrt{q}}[32]_{\sqrt{q}}[31]_{\sqrt{q}}[30]_{\sqrt{q}}[27]_{\sqrt{q}}[24]_{\sqrt{q}}[21]_{\sqrt{q}}[18]_{\sqrt{q}}[15]_{\sqrt{q}}[14]_{\sqrt{q}}}{[11]_{\sqrt{q}}[10]_{\sqrt{q}}[8]_{\sqrt{q}}[7]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[2]_{\sqrt{q}}}$$

$$D_q(6\,696\,000) = \frac{[33]_{\sqrt{q}}[32]_{\sqrt{q}}[31]_{\sqrt{q}}[30]_{\sqrt{q}}[25]_{\sqrt{q}}[24]_{\sqrt{q}}[21]_{\sqrt{q}}[20]_{\sqrt{q}}[15]_{\sqrt{q}}}{[11]_{\sqrt{q}}[10]_{\sqrt{q}}[7]_{\sqrt{q}}[6]_{\sqrt{q}}[5]_{\sqrt{q}}[4]_{\sqrt{q}}[3]_{\sqrt{q}}[2]_{\sqrt{q}}}.$$

Table A8. Casimir $Q(\Lambda)$ of **248**, **3875**, **27 000**, **30 380**, **147 250**, **779 247**, **1 763 125**, **2 450 240**, **4 096 000**, **4 881 384** and **6 696 000** for E_8.

Λ	**248**(Adj)	**3875**	**27 000**	**30 380**	**147 250**
$Q(\Lambda)$	30	48	62	60	72
779 247	**1 763 125**	**2 450 240**	**4 096 000**	**4 881 384**	**6 696 000**
80	96	90	93	100	98

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